

The Role of Expectations in a Macroeconomic Model with Inventories

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Abstract

In this paper we use a non-tâtonnement dynamic macroeconomic model with overlapping generations of consumers to study the role of expectations and inventories in the business cycle. Prices are fixed at the beginning of each period but adjusted between periods, taking into account possible market imbalances that have occurred within the period in an equilibrium with stochastic rationing. Producers hold inventories if they do not succeed to sell all their supply in the current period. Consumers too may store the consumption good so as to transfer it to the second period of their life. Whether they do this depends on their price expectations: only if they expect the price to rise will they desire to buy the planned consumption for both periods in the first period. Therefore price expectations are decisive for the type of dynamics that comes forth. In particular there are multiple equilibria in the sense that, for otherwise the same parameters but with different types of expectations, there are sequences of inflationary as well as deflationary equilibria with self-confirming expectations. In addition, and consistent with expectations, there may be endogenous expectations-switching along a trajectory. The above framework is applied to policy evaluations regarding the effectiveness of measures to overcome a quasi-stationary state of deflationary recession with underemployment, as is currently occurring in Japan. Such a state may have been provoked by a restrictive monetary shock and exasperated by over-investment and inventory holding, the latter by amplifying the spill-over effect from the goods to the labour market. If the recession is not too deep, creating inflationary expectations succeeds in exiting from the recession. Otherwise

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there may be a temporary effect of reducing unemployment but then the economy falls back into recession. Thus in that case other policy measures have to be taken, too. Among these, and contrary to conventional wisdom, balanced-budget cuts in taxes and government spending combined with downward rigidity of nominal wages seem to be the most effective ones.

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1 Introduction

2 The Model

We consider an economy in which there are n OLG-consumers, n' firms and a government. Consumers offer labor inelastically when young and consume a composite consumption good in both periods. They may buy this good in any period or in their first period of life only and transfer a part of it to the second period. The good is produced by firms using an atemporal production function whose only input is labor. Firms too may transfer unsold units of the consumption good into the future. The government levies a proportional tax on firms' profits to finance its expenditure for goods. Nevertheless, budget deficits and surpluses may arise and are made possible through money creation or destruction.

2.1 Timing of the Model

In period $t-1$ producers obtain an aggregate profit of Π_{t-1} which is distributed at the beginning of period t in part as tax to the government ($tax \Pi_{t-1}$) and in part to young consumers $((1 - tax) \Pi_{t-1})$, where $0 \leq tax \leq 1$. Also at the beginning of period t old consumers may hold a total quantity of money M_t , consisting of savings generated in period $t-1$. Thus households may use money as a means of transfer of purchasing power between periods. Whether they do this depends on their price expectations for their second period of life: since consumers may store consumption good bought in the first period, they will voluntarily hold money only if they expect the good's price to decrease. They may be forced, however, to do this in case they are rationed in their consumption goods purchases in the first period. Total money holdings in the economy at the end of period $t-1$ are $M_t + \Pi_{t-1}$.

Let X_t denote the aggregate quantity of the good purchased by young consumers in period t , p_t its price, w_t the nominal wage and L_t the aggregate quantity of labor. Then

$$M_{t+1} = (1 - tax) \Pi_{t-1} + w_t L_t - p_t X_t.$$

Denoting with G the quantity of goods purchased by the government and taking into account that old households want to consume all their money holdings in period t , the aggregate consumption of young and old households and of the government is $Y_t = X_t + \frac{M_t}{p_t} + G$. Using that $\Pi_t = p_t Y_t - w_t L_t$, considering $\Pi_t - \Pi_{t-1} = \Delta M_t^P$ as the variation in the money stock held by producers before they distribute profits and denoting with $\Delta M_t^C = M_{t+1} - M_t$ the one referring to consumers, we obtain $\Delta M_t^C + \Delta M_t^P = p_t G - tax \Pi_{t-1} = \text{budget deficit}$.

Denoting with S_t the aggregate amount of inventories carried over by firms to period t and with Y_t^p the aggregate amount of goods produced in period t , there results $S_{t+1} = Y_t^p + S_t - Y_t$.

2.2 The Consumption Sector

In his first period of life each consumer born at t is endowed with labor ℓ^s and an amount of money $(1 - tax) \Pi_{t-1}/n$ while his preferences are described by the utility function $u(x_t, x_{t+1}) = x_t^h x_{t+1}^{1-h}$, $0 < h < 1$, where x denotes consumption.¹ In solving his decision problem the young household has to decide whether to buy the quantities x_t and x_{t+1} in periods t and $t+1$, respectively, or buy the total quantity $x_t + x_{t+1}$ in period t and transfer x_{t+1} to period $t+1$. This in turn depends on the value of $\theta_t^e \equiv p_{t+1}^e/p_t$ where the superscript e stands for expectation. If $\theta_t^e < 1$, then the consumer expects a decrease in the goods price and hence prefers to buy x_{t+1} in his second period of life. In the opposite case $\theta_t^e > 1$ he buys everything in his first period.

We first treat the case $\theta_t^e < 1$. Then the consumer works with the budget constraints

$$0 \leq x_t \leq \omega_t^i, \quad 0 \leq x_{t+1} \leq (\omega_t^i - x_t) / \theta_t^e, \quad i = 0, 1$$

where

$$\omega_t^0 = \frac{1 - tax}{p_t} \frac{\Pi_{t-1}}{n} \quad \text{and} \quad \omega_t^1 = \omega_t^0 + \frac{w_t}{p_t} \ell^s$$

denote the consumer's real wealth when he is unemployed and employed, respectively. Implicit in this formulation is that rationing on the labor market is of the all-or-nothing type and that the labor market is visited before the goods market.

On the goods market the young household succeeds to buy its quantity demanded x_t^d with probability γ_t^d and is rationed to zero with probability $1 - \gamma_t^d$, where $\gamma_t^d \in [0, 1]$ is a rationing coefficient that the household perceives as given but that will be determined in equilibrium. Hence, the expected value of x_t is $\gamma_t^d x_t^d$, meaning that rationing is proportional and thus manipulable.

Effective demand x_t^{di} , $i = 0, 1$, is obtained by maximizing expected utility $\gamma_t^d x_t^h ((\omega_t^i - x_t) / \theta_t^e)^{1-h}$. The solution is $x_t^{di} = h \omega_t^i$. Thus the young consumer's effective demand is independent of γ_t^d and θ_t^e but it does depend on the real income ω_t^i and hence on whether the consumer has been employed.

¹See Colombo and Weinrich (2003b) for a more general approach to the consumer's problem.

Consider now the case $\theta_t^e > 1$. Then the consumer wants to buy the total quantity $x_t + x_{t+1} \equiv \widehat{x}_t$ in his first period of life and thus has to meet the budget constraint

$$x_t + x_{t+1} \leq \omega_t^i, \quad i = 0, 1.$$

Monotonicity of the utility function implies that his effective demand is $\widehat{x}_t^{di} = \omega_t^i$.

The aggregate supply of labor is $L^s = n\ell^s$. Denoting with L_t^d the aggregate demand of labor and with $\lambda_t^s = \min \left\{ \frac{L_t^d}{L^s}, 1 \right\}$ the fraction of young consumers that will be employed, the aggregate demand of goods by young consumers in case of deflationary expectations $\theta_t^e < 1$ is

$$\begin{aligned} X_t^d &= \lambda_t^s n x_t^{d1} + (1 - \lambda_t^s) n x_t^{d0} \\ &= h \left[(1 - tax) \frac{\Pi_{t-1}}{p_t} + \frac{w_t}{p_t} \lambda_t^s L^s \right] \equiv X^d \left(\lambda_t^s; \frac{w_t}{p_t}, \frac{(1 - tax) \Pi_{t-1}}{p_t}, h \right) \end{aligned} \quad (1)$$

whereas in case of inflationary expectations $\theta_t^e > 1$ it is

$$\begin{aligned} \widehat{X}_t^d &= \lambda_t^s n \widehat{x}_t^{d1} + (1 - \lambda_t^s) n \widehat{x}_t^{d0} \\ &= (1 - tax) \frac{\Pi_{t-1}}{p_t} + \frac{w_t}{p_t} \lambda_t^s L^s = X^d \left(\lambda_t^s; \frac{w_t}{p_t}, \frac{(1 - tax) \Pi_{t-1}}{p_t}, 1 \right). \end{aligned} \quad (2)$$

From (1) and (2) it is evident that the only difference in the aggregate effective demand by young consumers implied by different expectations $\theta_t^e < \text{or} > 1$ lies in the multiplicative factor $\tau \in \{h, 1\}$. We shall therefore identify the value of τ with the corresponding expectation type. Moreover, we shall assume $\tau = h$ in case $\theta_t^e = 1$.

The total effective aggregate demand of the consumption sector is now obtained by adding old consumers' aggregate demand $m_t = M_t/p_t$ and government demand G :

$$Y_t^d = X^d(\lambda_t^s; \alpha_t, (1 - tax) \pi_t, \tau) + m_t + G$$

where $\alpha_t \equiv w_t/p_t$ and $\pi_t \equiv \Pi_{t-1}/p_t$.

2.3 The Production Sector

Each of the n' identical firms uses an atemporal production function $y_t^p = f(\ell_t) = a\ell_t^b$, $a, b > 0$. Having transferred stocks from the previous period and being thus endowed with inventories s_t at the beginning of period t , the total amount supplied by a firm is $y_t^s = y_t^p + s_t$. As with consumers, firms too may be rationed, by means of a rationing mechanism analogous to that assumed for the consumption sector.

Denoting the single firm's effective demand of labor by ℓ_t^d , the quantity of labor effectively transacted is ℓ_t^d with probability λ_t^d and 0 with probability $1 - \lambda_t^d$, where $\lambda_t^d \in [0, 1]$. It is obvious that $E\ell_t = \lambda_t^d \ell_t^d$. On the goods market the rationing rule is assumed to be

$$y_t = \begin{cases} y_t^s, & \text{with prob. } \sigma \gamma_t^s \\ d_t y_t^s, & \text{with prob. } 1 - \sigma \gamma_t^s \end{cases},$$

where $\sigma \in (0, 1)$, $\gamma_t^s \in [0, 1]$ and $d_t = (\gamma_t^s - \sigma \gamma_t^s) / (1 - \sigma \gamma_t^s)$. σ is a fixed parameter of the mechanism whereas λ_t^d and γ_t^s are perceived rationing coefficients taken as given by the firm the effective value of which will be determined in equilibrium. The definition of d_t implies that $E y_t = \gamma_t^s y_t^s$ which, in particular, is independent of σ . It is obvious that $E \ell_t = \lambda_t^d \ell_t^d$.

The firm's effective demand $\ell_t^d = \ell^d(\gamma_t^s; \alpha_t)$ is obtained from maximizing its expected profit $\gamma_t^s [f(\ell_t^d) + s_t] - \alpha_t \ell_t^d$ subject to

$$0 \leq \ell_t^d \leq \frac{d_t}{\alpha_t} [f(\ell_t^d) + s_t]$$

while its effective supply is $y_t^s = f(\ell_t^d) + s_t$. The upper bound on labor demand reflects the fact that the firm must be prepared to finance labor service purchases even if rationed on the goods market (since the labor market is visited first, it will know whether it is rationed on the goods market only after it has hired labor). In general the solution depends on this constraint but it is not binding (see Appendix 1, Lemma A.1) if we make the assumption $b \leq 1 - \sigma$. In this case, labor demand is

$$\ell_t^d = \ell^d(\gamma_t^s; \alpha_t) = \left(\frac{\gamma_t^s a b}{\alpha_t} \right)^{\frac{1}{1-b}}. \quad (3)$$

Notice that labor demand is independent of s_t . The aggregate labor demand then is $L_t^d = n' \ell^d(\gamma_t^s; \alpha_t) \equiv L^d(\gamma_t^s; \alpha_t)$ and, because only a fraction λ_t^d of firms can hire workers, the aggregate supply of goods is

$$Y_t^s = \lambda_t^d n' f(\ell^d(\gamma_t^s; \alpha_t)) + S_t \equiv Y^s(\lambda_t^d, \gamma_t^s; \alpha_t, S_t). \quad (4)$$

3 Temporary Equilibrium Allocations

For any given period t we can now describe a feasible allocation as a temporary equilibrium with rationing as follows.

Definition 1 *Given a real wage α_t , a real profit level π_t , real money balances m_t , inventories S_t , a level of public expenditure G , a tax rate τ and an expectation type $\tau \in \{h, 1\}$, a list of rationing coefficients $(\gamma_t^d, \gamma_t^s, \lambda_t^d, \lambda_t^s, \delta_t, \varepsilon_t) \in [0, 1]^6$ and an aggregate allocation (\bar{L}_t, \bar{Y}_t) constitute a temporary equilibrium if the following conditions are fulfilled:*

- (C1) $\bar{L}_t = \lambda_t^s L^s = \lambda_t^d L^d(\gamma_t^s; \alpha_t)$;
- (C2) $\bar{Y}_t = \gamma_t^s Y^s(\lambda_t^d, \gamma_t^s; \alpha_t, S_t) = \gamma_t^d X^d(\lambda_t^s; \alpha_t, (1 - \tau)\pi_t, \tau) + \delta_t m_t + \varepsilon_t G$;
- (C3) $(1 - \lambda_t^s)(1 - \lambda_t^d) = 0$; $(1 - \gamma_t^s)(1 - \gamma_t^d) = 0$;
- (C4) $\gamma_t^d(1 - \delta_t) = 0$; $\delta_t(1 - \varepsilon_t) = 0$.

Conditions (C1) and (C2) require that expected aggregate transactions balance. This means that agents have correct perceptions of the rationing coefficients $\gamma_t^d, \gamma_t^s, \lambda_t^d$ and λ_t^s . Equations (C3) formalize the short-side rule according to which at most one side on each market is rationed. The meaning of the coefficients δ_t and ε_t in (C2) and (C4) is that also old households and/or the government can be rationed. However, according to condition (C4) this may occur only after young households have been rationed (to zero).

As shown in the table below it is possible to distinguish different types of equilibrium according to which market sides are rationed: excess supply on both markets is called *Keynesian Unemployment* [K], excess demand on both markets *Repressed Inflation* [I], excess supply on the labor market and excess demand on the goods market *Classical Unemployment* [C] and excess demand on the labor market with excess supply on the goods market *Underconsumption* [U].

	K	I	C	U
λ_t^s	< 1	= 1	< 1	= 1
λ_t^d	= 1	< 1	= 1	< 1
γ_t^s	< 1	= 1	= 1	< 1
γ_t^d	= 1	< 1	< 1	= 1
δ_t	= 1	≤ 1	≤ 1	= 1
ε_t	= 1	≤ 1	≤ 1	= 1

Of course there are further intermediate cases which, however, can be considered as limiting cases of the above ones. In particular, when all the rationing coefficients are equal to one, we are in a Walrasian Equilibrium.²

Existence and uniqueness of temporary equilibrium are established by the following proposition.

Proposition 1 *For any quadruple of variables $(\alpha_t, m_t, \pi_t, S_t)$, with α_t strictly positive and m_t, π_t and S_t non-negative, any non-negative pair of policy parameters (G, tax) and any expectation type $\tau \in \{h, 1\}$ there exists a unique temporary equilibrium allocation (\bar{L}_t, \bar{Y}_t) . \bar{L}_t is given by*

$$\bar{L}_t = \min \left\{ \tilde{L}(\alpha_t, \pi_t, m_t, S_t, G, tax, \tau), L^d(1, \alpha_t), L^s \right\} \equiv \mathcal{L}(\alpha_t, \pi_t, m_t, S_t, G, tax, \tau) \quad (5)$$

where $\tilde{L}(\alpha_t, \pi_t, m_t, S_t, G, tax, \tau)$ is the unique solution in L of

$$\alpha_t \left(\frac{1}{b} - \tau \right) L + \frac{\alpha_t}{ab} \left(\frac{L}{n'} \right)^{1-b} S_t = \tau (1 - tax) \pi_t + m_t + G \quad (6)$$

and

$$L^d(1, \alpha_t) = n' \left(\frac{ab}{\alpha_t} \right)^{\frac{1}{1-b}}. \quad (7)$$

²For an illustration of equilibrium regimes and their representation in the $p - w$ plane, see Colombo and Weinrich (2003b).

$\bar{Y}_t \equiv \mathcal{Y}(\alpha_t, \pi_t, m_t, S_t, G, tax, \tau)$ is determined as follows. If $\bar{L}_t = \tilde{L}(\cdot)$, then $\bar{Y}_t = \frac{\alpha_t}{b} \bar{L}_t + \frac{\alpha_t}{ab} \left(\frac{\bar{L}_t}{n^t} \right)^{1-b} S_t$, and if $\bar{L}_t = L^d(1, \alpha_t)$, then $\bar{Y}_t = \frac{\alpha_t}{b} L^d(1, \alpha_t) + S_t$. Finally, if $\bar{L}_t = L^s$, then $\bar{Y}_t = \min \left\{ \frac{\alpha_t}{b} L^s + S_t, \tau(1 - tax) \pi_t + \tau \alpha_t L^s + m_t + G \right\}$.

Proof. See Appendix 2.

For the sake of illustration let us consider a situation of Keynesian Unemployment. This type of equilibrium involves rationing of households on the labor market and of firms on the goods market. It is given by a pair $(\lambda_t^s, \gamma_t^s)$ such that

$$\begin{aligned} \bar{L}_t &= \lambda_t^s L^s = L^d(\gamma_t^s) \\ \bar{Y}_t &= \gamma_t^s Y^s(1, \gamma_t^s) = X^d(\lambda_t^s) + m_t + G \end{aligned}$$

(where we have suppressed all arguments that are not rationing coefficients).

The consumption sector supplies the amount of labor $L^s > \bar{L}_t$ and demands the quantity of goods $Y_t^d = \bar{Y}_t$ whereas firms demand labor $L_t^d = \bar{L}_t$ and supply $Y_t^s > \bar{Y}_t$ of goods. It follows that $\lambda_t^s = \bar{L}_t / L^s$, $\gamma_t^s = \bar{Y}_t / Y_t^s$ and $\lambda_t^d = \gamma_t^d = 1$ ($= \delta_t = \varepsilon_t$), which are just the values that led households and firms to express their respective transaction offers. Thus their expectations regarding these rationing coefficients are confirmed. Nevertheless, due to the randomness in rationing at an individual agent's level, effective aggregate demands and supplies of rationed agents exceed their actual transactions. Moreover, as indicated earlier, these excesses can be used to get an indicator of the strength of rationing. Since there is zero-one rationing on the labor market, $1 - \lambda_t^s = (L^s - \bar{L}_t) / L^s$ is the ratio of the number of unemployed workers and the total number of young households. Regarding the goods market, in a K -equilibrium $\bar{Y}_t - \gamma_t^s Y^s(1, \gamma_t^s) = 0$, and therefore

$$\frac{d(1 - \gamma_t^s)}{d\bar{Y}_t} = -\frac{1}{Y_t^s + \gamma_t^s \frac{\partial Y^s}{\partial \gamma_t^s}} < 0$$

since $\frac{\partial Y^s}{\partial \gamma_t^s}(1, \gamma_t^s) = n^t f'(\ell^d(\gamma_t^s)) \frac{d\ell^d}{d\gamma_t^s} > 0$. So a decrease in \bar{Y}_t (for example due to a reduction in government spending), and thus an aggravation of the shortage of aggregate demand for firms' goods, is unambiguously related to an increase in $1 - \gamma_t^s$ which can therefore be interpreted as a measure of the strength of rationing on the goods market. A similar reasoning justifies the use as rationing measures of the terms $1 - \lambda_t^d$ and $1 - \gamma_t^d$ in the other equilibrium regimes.

4 Dynamics

So far our analysis has been essentially static. For any $(\alpha_t, \pi_t, m_t, S_t)$, (G, tax) and τ we have described a feasible allocation in terms of a temporary equilibrium with rationing. To extend now our analysis to a dynamic one we must link successive periods one to another. This link will be given by the adjustment

of prices, by the changes in the stock of money and in profits and by possible changes in the expectation type. The latter is a somewhat subtle issue which we will treat as the last point in our description of the dynamics. For the moment we assume a given expectation type $\tau \in \{h, 1\}$ and proceed as if this type were constant. Later we will introduce the possibility of expectation switching.

For given τ , the dynamics in profits, money and inventories follow from the definition of these variables and equations (5) to (7), i.e.

$$\Pi_t = p_t \bar{Y}_t - w_t \bar{L}_t,$$

$$\begin{aligned} M_{t+1} &= (1 - tax) \Pi_{t-1} + w_t \bar{L}_t - p_t \bar{Y}_t + \delta_t M_t + \varepsilon_t p_t G \\ &= (1 - tax) \Pi_{t-1} - \Pi_t + \delta_t M_t + \varepsilon_t p_t G, \end{aligned}$$

$$S_{t+1} = Y^s(\lambda_t^d, \gamma_t^s; \alpha_t, S_t) - \bar{Y}_t$$

where

$$\bar{Y}_t = \mathcal{Y}(\alpha_t, \pi_t, m_t, S_t, G, tax, \tau) \text{ and } \bar{L}_t = \mathcal{L}(\alpha_t, \pi_t, m_t, S_t, G, tax, \tau).$$

As for the adjustment of prices and wages we assume that, whenever an excess of demand (supply) is observed, the price rises (falls). In terms of the rationing coefficients observed in period t , this amounts to

$$p_{t+1} < p_t \Leftrightarrow \gamma_t^s < 1; \quad p_{t+1} > p_t \Leftrightarrow \gamma_t^d < 1,$$

$$w_{t+1} < w_t \Leftrightarrow \lambda_t^s < 1; \quad w_{t+1} > w_t \Leftrightarrow \lambda_t^d < 1.$$

More precisely, in our simulation model we have specified these adjustments as follows:

$$p_{t+1} = \begin{cases} [1 - \mu_1(1 - \gamma_t^s)] p_t & \text{if } \gamma_t^s < 1 \\ \left[1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)\right] p_t & \text{if } \gamma_t^d < 1 \end{cases} \quad (8)$$

$$w_{t+1} = \begin{cases} [1 - \nu_1(1 - \lambda_t^s)] w_t & \text{if } \lambda_t^s < 1 \\ [1 + \nu_2(1 - \lambda_t^d)] w_t & \text{if } \lambda_t^d < 1 \end{cases} \quad (9)$$

where $\mu_1, \mu_2, \nu_1, \nu_2 \in [0, 1]$. Then the adjustment equations for the real wage are

$$\alpha_{t+1} = \begin{cases} \frac{1 - \nu_1(1 - \lambda_t^s)}{1 - \mu_1(1 - \gamma_t^s)} \alpha_t & \text{if } (\bar{L}_t, \bar{Y}_t) \in K \\ \frac{1 - \nu_1(1 - \lambda_t^s)}{1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)} \alpha_t & \text{if } (\bar{L}_t, \bar{Y}_t) \in C \\ \frac{1 + \nu_2(1 - \lambda_t^d)}{1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)} \alpha_t & \text{if } (\bar{L}_t, \bar{Y}_t) \in I \\ \frac{1 + \nu_2(1 - \lambda_t^d)}{1 - \mu_1(1 - \gamma_t^s)} \alpha_t & \text{if } (\bar{L}_t, \bar{Y}_t) \in U \end{cases} \quad (10)$$

whereas the inflation factor $\theta_t = p_{t+1}/p_t$ is given by

$$\theta_t = \begin{cases} 1 - \mu_1 (1 - \gamma_t^s) & \text{if } (\bar{L}_t, \bar{Y}_t) \in K \cup U \\ 1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right) & \text{if } (\bar{L}_t, \bar{Y}_t) \in C \cup I \end{cases}. \quad (11)$$

The dynamics of the model in real terms is then given by the sequence $\{(\alpha_t, m_t, \pi_t, S_t)\}_{t=1}^\infty$, where α_{t+1} is as in (10),

$$\pi_{t+1} = \frac{\bar{Y}_t - \alpha_t \bar{L}_t}{\theta_t},$$

$$m_{t+1} = \frac{1}{\theta_t} [\delta_t m_t + \varepsilon_t G + (1 - tax) \pi_t] - \pi_{t+1}$$

and

$$S_{t+1} = \lambda_t^d n' a \left(\frac{\gamma_t^{s ab}}{\alpha_t} \right)^{\frac{b}{1-b}} + S_t - \bar{Y}_t.$$

What has still to be determined here are the values of the rationing coefficients $(\gamma_t^d, \gamma_t^s, \lambda_t^d, \lambda_t^s, \delta_t, \varepsilon_t)$. This will be done in Appendix 3 where we will also give the corresponding explicit equations of the complete dynamic system.

We introduce now the possibility of expectation switching. We would like this to occur whenever it is required in order to keep expectations correct along a trajectory of the system. For example, consider the case that, in period t , consumers have deflationary expectations ($\theta_t^e \leq 1$ or, equivalently, $\tau = h$) but the equilibrium in period t is such that there is excess demand on the goods market and thus $p_{t+1} > p_t$. Then the assumption $\tau = h$ in period t has been incorrect and we substitute it by $\tau = 1$, i.e. $\theta_t^e > 1$. Of course then a different equilibrium arises in period t but we claim that the type of equilibrium is still such that there is excess demand on the goods market. Therefore expectations have been adjusted so as to become correct. Analogously we correct the expectations in case $\theta_t^e > 1$ but the equilibrium in period t involves excess supply on the goods market. The rationale for doing this is given by the following

Lemma 2 *Assume that for $\tau = h$ in period t an equilibrium with $\gamma_t^d < 1$ occurs. Then this inequality is preserved when switching in period t to $\tau = 1$. Conversely, assume that for $\tau = 1$ in period t an equilibrium with $\gamma_t^s < 1$ occurs. Then this inequality is preserved when switching in period t to $\tau = h$.*

Proof: Assume $\tau = h$ in period t and in the corresponding equilibrium we have $\gamma_t^d < 1$. Then there is excess demand on the goods market, $Y_t^d = X_t^d + m_t + G > Y_t^s$. If τ is changed to $\tau = 1$, then by (1) and (2) X_t^d increases. Thus the excess of demand over supply on the goods market can only increase and in particular $Y_t^d > Y_t^s$ still holds.

Conversely, consider $\tau = 1$ in period t and $\gamma_t^s < 1$. Then $Y_t^d < Y_t^s$ and changing τ from 1 to h decreases X_t^d , thus Y_t^d , and $Y_t^d < Y_t^s$ is preserved. ■

Taking into account expectations switching a trajectory of the dynamic system is given by a sequence $\{(\alpha_t, m_t, \pi_t, S_t, \tau_t)\}_{t=1}^{\infty}$.

5 Simulations

6 Policy and the Japanese Deflationary Recession

7 Conclusions

References

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Appendix 1: Lemma A.1

Lemma A.1 *When $b \leq 1 - \sigma$, the solution to the firm's maximization problem is independent of the constraint $\ell_t^d \leq \frac{d_t}{\alpha_t} [f(\ell_t^d) + s_t]$.*

Proof. The first order condition for an interior solution of the firm's problem is

$$\gamma^s f'(\ell) = \alpha \Leftrightarrow \gamma^s \frac{bf(\ell)}{\ell} = \alpha \Leftrightarrow \ell = \gamma^s \frac{bf(\ell)}{\alpha}.$$

Moreover the inequalities $\frac{1}{b} \geq \frac{1}{1-\sigma} \geq \frac{1-\gamma^s\sigma}{1-\sigma}$ yield $1 \leq \frac{1-\sigma}{b(1-\gamma^s\sigma)}$. From this follows

$$\ell \leq \frac{\gamma^s(1-\sigma)}{1-\gamma^s\sigma} \frac{1}{\gamma^s} \frac{1}{b} \ell = d \frac{1}{\gamma^s} \frac{1}{b} \ell = d \frac{1}{\gamma^s} \frac{1}{b} \gamma^s \frac{bf(\ell)}{\alpha} = \frac{d}{\alpha} f(\ell),$$

which proves our claim. ■

Appendix 2: Proof of Proposition 1

Since we hold $\{\alpha_t, m_t, \pi_t, S_t\}$, (G, tax) and τ fixed, we omit these variables whenever possible as arguments in the subsequent functions. Define the set

$$\overline{H} \equiv \left\{ \left(\lambda^s L^s, \gamma^d X^d(\lambda^s) \right) \mid \left(\lambda^s, \gamma^d \right) \in [0, 1]^2 \right\}$$

and its subsets $\overline{H}^K = \overline{H} \mid_{\gamma^d=1, \lambda^s < 1}$, $\overline{H}^I = \overline{H} \mid_{\gamma^d < 1, \lambda^s=1}$, $\overline{H}^C = \overline{H} \mid_{\gamma^d < 1, \lambda^s < 1}$ and $\overline{H}^U = \overline{H} \mid_{\gamma^d=1, \lambda^s=1}$. Using the terminology introduced by Honkapohja and Ito (1985), we derive from these the consumption sector's trade curves

$$\overline{H}_0^K = \overline{H}^K + \{(0, m_t + G)\} = \left\{ \left(\lambda^s L^s, X^d(\lambda^s) + m_t + G \right) \mid \lambda^s \in [0, 1] \right\},$$

$$\begin{aligned} \overline{H}_0^I = & \left\{ \left(L^s, \gamma^d X^d(1) + m_t + G \right) \mid \gamma^d \in (0, 1) \right\} \cup \{(L^s, \delta m_t + G) \mid \delta \in (0, 1]\} \\ & \cup \{(L^s, \varepsilon G) \mid \varepsilon \in [0, 1]\}, \end{aligned}$$

$$\overline{H}_0^C = \left\{ \left(\lambda^s L^s, \gamma^d X^d(\lambda^s) + m_t + G \right) \mid \left(\lambda^s, \gamma^d \right) \in [0, 1] \times (0, 1) \right\}$$

$$\cup \{(\lambda^s L^s, \delta m_t + G) \mid (\lambda^s, \delta) \in [0, 1] \times (0, 1]\} \cup \{(\lambda^s L^s, \varepsilon G) \mid (\lambda^s, \varepsilon) \in [0, 1] \times [0, 1]\}.$$

and

$$\overline{H}_0^U = \overline{H}^U + \{(0, m_t + G)\} = \left\{ \left(L^s, X^d(1) + m_t + G \right) \right\}.$$

Similarly, starting from

$$\overline{F} \equiv \left\{ \left(\lambda^d L^d(\gamma^s), \gamma^s Y^s(\lambda^d, \gamma^s) \right) \mid \left(\lambda^d, \gamma^s \right) \in [0, 1]^2 \right\}$$

we define the production sector's trade curves as $\overline{F}^K = \overline{F} \mid_{\lambda^d=1, \gamma^s < 1}$, $\overline{F}^I = \overline{F} \mid_{\lambda^d < 1, \gamma^s=1}$, $\overline{F}^C = \overline{F} \mid_{\lambda^d=1, \gamma^s=1}$ and $\overline{F}^U = \overline{F} \mid_{\lambda^d < 1, \gamma^s < 1}$. To derive them, we begin with noticing that

$$\gamma^s Y^s(\lambda^d, \gamma^s; \alpha_t, S_t) = \frac{\alpha_t}{b} \lambda^d L^d(\gamma_t^s; \alpha_t) + \gamma^s S_t. \quad (\text{A.1})$$

Indeed, by (4)

$$\gamma^s Y^s \left(\lambda^d, \gamma^s; \alpha_t, S_t \right) = \gamma^s \left[\lambda^d n' f \left(\ell^d (\gamma_t^s; \alpha_t) \right) + S_t \right]$$

whereas from $f(\ell) = a\ell^b$ follows $f'(\ell) = b\frac{f(\ell)}{\ell}$, which implies $f(\ell) = \frac{1}{b}f'(\ell)\ell$. Therefore

$$\gamma^s Y^s \left(\lambda^d, \gamma^s; \alpha_t, S_t \right) = \gamma^s \left[\lambda^d n' \frac{1}{b} f' \left(\ell^d (\gamma_t^s; \alpha_t) \right) \ell^d (\gamma_t^s; \alpha_t) + S_t \right].$$

But $\gamma^s f'(\ell^d(\gamma_t^s; \alpha_t)) = \alpha_t$ from any producer's optimizing behavior, and thus

$$\gamma^s Y^s \left(\lambda^d, \gamma^s; \alpha_t, S_t \right) = \frac{\alpha_t}{b} \lambda^d n' \ell^d (\gamma_t^s; \alpha_t) + \gamma^s S_t = \frac{\alpha_t}{b} \lambda^d L^d (\gamma_t^s; \alpha_t) + \gamma^s S_t.$$

This implies immediately that

$$\overline{F}^C = \left\{ \left(L^d(1; \alpha_t), \frac{\alpha_t}{b} L^d(1; \alpha_t) + S_t \right) \right\}.$$

Consider now

$$\overline{F}^K = \left\{ \left(L^d(\gamma^s; \alpha_t), \gamma^s Y^s(1, \gamma^s; \alpha_t, S_t) \right) \mid \gamma^s \in [0, 1) \right\}.$$

Then (A.1) yields

$$\gamma^s Y^s(1, \gamma^s; \alpha_t, S_t) = \frac{\alpha_t}{b} L^d(\gamma_t^s; \alpha_t) + \gamma^s S_t.$$

On the other hand, (3) implies

$$\gamma^s = \frac{\alpha_t}{ab} \left(\ell^d(\gamma_t^s; \alpha_t) \right)^{1-b} = \frac{\alpha_t}{ab} \left(\frac{L^d(\gamma_t^s; \alpha_t)}{n'} \right)^{1-b}$$

and therefore

$$\gamma^s Y^s(1, \gamma^s; \alpha_t, S_t) = \frac{\alpha_t}{b} L^d(\gamma_t^s; \alpha_t) + \frac{\alpha_t}{ab} \left(\frac{L^d(\gamma_t^s; \alpha_t)}{n'} \right)^{1-b} S_t.$$

Since $L^d(\gamma_t^s; \alpha_t)$ is strictly increasing in γ_t^s , this yields

$$\overline{F}^K = \left\{ \left(L, \frac{\alpha_t}{b} L + \frac{\alpha_t}{ab} \left(\frac{L}{n'} \right)^{1-b} S_t \right) \mid 0 \leq L < L^d(1; \alpha_t) \right\}. \quad (\text{A.2})$$

Consider next

$$\overline{F}^I = \left\{ \left(\lambda^d L^d(1; \alpha_t), Y^s(\lambda^d, 1; \alpha_t, S_t) \right) \mid \lambda^d \in [0, 1) \right\}.$$

By (A.1) $Y^s(\lambda^d, 1; \alpha_t) = \frac{\alpha_t}{b} \lambda^d L^d(1; \alpha_t) + S_{t-1}$ and therefore

$$\overline{F}^I = \left\{ \left(L, \frac{\alpha_t}{b} L + S_t \right) \mid 0 \leq L < L^d(1; \alpha_t) \right\}.$$

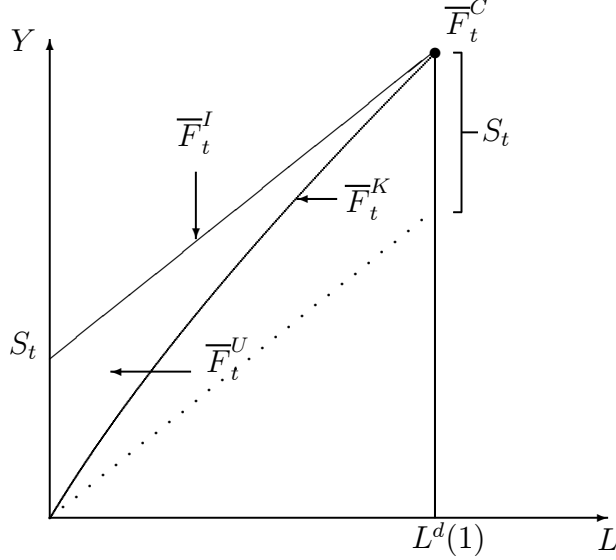


Figure A.1: The producers' trade curves

Since $\frac{\alpha_t}{ab} \left(\frac{L}{n'}\right)^{1-b} = \gamma^s \leq 1$, \overline{F}^K is positioned below \overline{F}^I .

Finally consider \overline{F}^U . It is given by

$$\overline{F}^U = \left\{ \left(\lambda^d L^d(\gamma^s; \alpha_t), \frac{\alpha_t}{b} \lambda^d L^d(\gamma_t^s; \alpha_t) + \frac{\alpha_t}{ab} \left(\frac{L^d(\gamma_t^s; \alpha_t)}{n'} \right)^{1-b} S_t \right) \mid (\lambda^d, \gamma^s) \in [0, 1)^2 \right\} \quad (\text{A.3})$$

Comparing with \overline{F}^K and \overline{F}^I , it is clear that \overline{F}^U is the set of points contained between \overline{F}^K and \overline{F}^I . Figure A.1 illustrates the producers' trade curves.

Using the consumption sector's and the production sector's trade curves and indicating with S^c the closure of the set S , we now note that a pair $(\overline{L}, \overline{Y}) \in R_+^2$ is a temporary equilibrium allocation if and only if it is an element of the set

$$Z = \left((\overline{H}_0^K)^c \cap (\overline{F}^K)^c \right) \cup \left((\overline{H}_0^I)^c \cap (\overline{F}^I)^c \right) \cup \left((\overline{H}_0^C)^c \cap (\overline{F}^C)^c \right) \cup \left((\overline{H}_0^U)^c \cap (\overline{F}^U)^c \right).$$

To show existence of an equilibrium is equivalent to showing that Z is not empty. To this end consider first the locus

$$(\overline{H}_0^K)^c = \left\{ \left(\lambda_t^s L^s, X^d(\lambda_t^s) + m_t + G \right) \mid \lambda_t^s \in [0, 1] \right\}$$

and recall that

$$X^d(\lambda_t^s) = \tau [(1 - tax) \pi_t + \alpha_t \lambda_t^s L^s].$$

Defining the function

$$\Gamma_t(L) = \tau [(1 - tax) \pi_t + \alpha_t L] + m_t + G, \quad L \geq 0,$$

we see that $(\overline{H}_0^K)^c$ is the part of the graph of Γ_t for which $L \leq L^s$.

Next consider again the production sector's trade curves. From (A.2) we conclude that the locus $(\overline{F}^K)^c$ is the part of the graph of the function

$$\Delta_t(L) = \frac{\alpha_t}{b}L + \frac{\alpha_t}{ab} \left(\frac{L}{n'} \right)^{1-b} S_t, \quad L \geq 0,$$

for which $L \leq L^d(1)$. Notice that the graphs of the functions Γ_t and Δ_t always intersect. Indeed, $\Gamma'_t(L) = \tau\alpha_t$ and $\Gamma_t(0) = \tau(1 - tax)\pi_t + m_t + G > 0$, whereas $\Delta'_t(L) \geq \frac{\alpha_t}{b} > \tau\alpha_t$ (since $1/b > 1 \geq \tau$) and $\Delta_t(0) = 0$. Setting $\Delta_t(L) = \Gamma_t(L)$ yields (6) with the unique solution denoted $\tilde{L}(\alpha_t, \pi_t, m_t, G, tax, \tau)$. Therefore the equilibrium level on the labor market is

$$\overline{L}_t = \min \left\{ \tilde{L}(\alpha_t, \pi_t, m_t, G, tax, \tau), L^d(1, \alpha_t), L^s \right\} = \mathcal{L}(\alpha_t, \pi_t, m_t, S_t, G, tax, \tau)$$

whereas the one on the goods market is, by definition of the function $\mathcal{Y}(\cdot)$,

$$\overline{Y}_t = \mathcal{Y}(\alpha_t, \pi_t, m_t, S_t, G, tax, \tau).$$

This shows that the equilibrium allocation

$$(\overline{L}_t, \overline{Y}_t) = (\mathcal{L}(\alpha_t, \pi_t, m_t, S_t, G, tax, \tau), \mathcal{Y}(\alpha_t, \pi_t, m_t, S_t, G, tax, \tau))$$

exists and is uniquely defined. ■

Appendix 3: The explicit complete dynamic system

The dynamic system is given by four different subsystems, one for each of the equilibrium types K , I , C and U , and endogenous regime switching. For given (G, tax) and $\tau \in \{h, 1\}$, any list $(\alpha_t, \pi_t, m_t, S_t)$ gives rise to a uniquely determined equilibrium allocation $(\overline{L}_t, \overline{Y}_t)$ being of one of the above types (or of an intermediate one). More precisely, equation (5) allows us to characterize the type of equilibrium defined in Table 1: if $\overline{L}_t = \tilde{L}(\alpha_t, \pi_t, m_t, S_t, G, tax, \tau)$, the resulting equilibrium is of type K or a limiting case of it. If $\overline{L}_t = L^d(1, \alpha_t)$, type C or a limiting case of it occurs. Finally, if $\overline{L}_t = L^s$, an equilibrium of type I or a limiting case results if $\frac{\alpha_t}{b}L^s + S_t \leq \tau(1 - tax)\pi_t + \tau\alpha_t L^s + m_t + G$; otherwise the equilibrium is of type U . Regime switching may occur because $(\overline{L}_t, \overline{Y}_t)$ may be of type $T \in \{K, I, C, U\}$ and $(\overline{L}_{t+1}, \overline{Y}_{t+1})$ of type $T' \neq T$.

The above discussion and Proposition 1 allow us to determine the expressions of those rationing coefficients which are possibly smaller than one. This is summarized in the following corollary of Proposition 1.

Corollary A.1 *In case K , $\lambda_t^s = \frac{\overline{L}_t}{L^s}$ and $\gamma_t^s = \frac{\alpha_t}{ab} \left(\frac{\overline{L}_t}{n'} \right)^{1-b}$. In case C , $\lambda_t^s = \frac{\overline{L}_t}{L^s}$ and, in case I , $\lambda_t^d = \frac{L^s}{L^d(1, \alpha_t)}$. Moreover, in both these latter cases,*

$$(\gamma_t^d, \delta_t, \varepsilon_t) = \begin{cases} \left(\frac{\overline{Y}_t - m_t - G}{\tau(1 - tax)\pi_t + \tau\alpha_t \overline{L}_t}, 1, 1 \right) & \text{if } \overline{Y}_t \geq G + m_t \\ \left(0, \frac{\overline{Y}_t - G}{m_t}, 1 \right) & \text{if } G + m_t > \overline{Y}_t \geq G \\ \left(0, 0, \frac{\overline{Y}_t}{G} \right) & \text{if } \overline{Y}_t < G \end{cases}$$

Finally, in case U $\gamma_t^s = \frac{1}{S_t} (\bar{Y}_t - \frac{\alpha_t}{b} \bar{L}_t)$ and $\lambda_t^d = \bar{L}_t / L^d(\gamma_t^s; \alpha_t)$.

Proof. We start with case U . Then, by (A.3) it must be true that

$$(\bar{L}_t, \bar{Y}_t) = \left(\lambda^d L^d(\gamma_t^s; \alpha_t), \frac{\alpha_t}{b} \lambda^d L^d(\gamma_t^s; \alpha_t) + \frac{\alpha_t}{ab} \left(\frac{L^d(\gamma_t^s; \alpha_t)}{n'} \right)^{1-b} S_t \right).$$

Moreover by (3)

$$L^d(\gamma_t^s; \alpha_t) = n' \left(\frac{\gamma_t^s ab}{\alpha_t} \right)^{\frac{1}{1-b}}.$$

Therefore

$$\frac{\alpha_t}{b} \lambda^d L^d(\gamma_t^s; \alpha_t) + \frac{\alpha_t}{ab} \left(\frac{L^d(\gamma_t^s; \alpha_t)}{n'} \right)^{1-b} S_t = \bar{Y}_t$$

\Leftrightarrow

$$\frac{\alpha_t}{b} \lambda_t^d L^d(\gamma_t^s; \alpha_t) + \gamma_t^s S_t = \bar{Y}_t$$

Recalling that $\lambda^d L^d(\gamma_t^s; \alpha_t) = \bar{L}_t$ and solving for γ_t^s yields the claimed expression.

In all cases, the values of λ_t^s and λ_t^d are immediate by definition. The value of γ_t^s in case K can be obtained using equation (3). Finally, γ_t^d , δ_t , ε_t are determined by means of Definition 1 and equations (1) and (2). ■

We can now give the explicit equations of all subsystems of the dynamical system.

Keynesian unemployment system

Employment level: $\bar{L}_t = \tilde{L}(\alpha_t, \pi_t, m_t, S_t, G, tax, \tau)$.

Output level: $\bar{Y}_t = \frac{\alpha_t}{b} \bar{L}_t + \frac{\alpha_t}{ab} \left(\frac{\bar{L}_t}{n'} \right)^{1-b} S_t$.

Rationing coefficients: $\lambda_t^s = \frac{\bar{L}_t}{L^s}$, $\lambda_t^d = 1$, $\gamma_t^s = \frac{\alpha_t}{ab} \left(\frac{\bar{L}_t}{n'} \right)^{1-b}$, $\gamma_t^d = 1$, $\delta_t = \varepsilon_t = 1$.

Price inflation: $\theta_t = 1 - \mu_1 (1 - \gamma_t^s)$.

Real wage adjustment: $\alpha_{t+1} = \frac{1-\nu_1(1-\lambda_t^s)}{1-\mu_1(1-\gamma_t^s)} \alpha_t$.

Real profit: $\pi_{t+1} = \frac{1}{\theta_t} (\bar{Y}_t - \alpha_t \bar{L}_t)$.

Real money stock: $m_{t+1} = \frac{1}{\theta_t} [m_t + G + (1 - tax) \pi_t] - \pi_{t+1}$.

Inventories: $S_{t+1} = n' a \left(\frac{ab \gamma_t^s}{\alpha_t} \right)^{\frac{b}{1-b}} + S_t - \bar{Y}_t$.

Repressed inflation system

$\bar{L}_t = L^s$.

$\bar{Y}_t = \frac{\alpha_t}{b} \bar{L}_t + S_t$.

$\lambda_t^s = 1$, $\lambda_t^d = \frac{L^s}{L^d(1, \alpha_t)}$; $\gamma_t^s = 1$.

If $\bar{Y}_t \geq G + m_t$, then $\gamma_t^d = \frac{\bar{Y}_t - m_t - G}{\tau(1-tax)\pi_t + \tau\alpha_t \bar{L}_t}$, $\delta_t = \varepsilon_t = 1$;

if $G + m_t > \bar{Y}_t \geq G$, then $\gamma_t^d = 0$, $\delta_t = \frac{\bar{Y}_t - G}{m_t}$, $\varepsilon_t = 1$;

if $\bar{Y}_t < G$, then $\gamma_t^d = \delta_t = 0$, $\varepsilon_t = \frac{\bar{Y}_t}{G}$.

$$\begin{aligned}
\theta_t &= 1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3} \right). \\
\alpha_{t+1} &= \frac{1 + \nu_2(1 - \lambda_t^d)}{1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3} \right)} \alpha_t. \\
\pi_{t+1} &= \frac{1}{\theta_t} (\bar{Y}_t - \alpha_t \bar{L}_t). \\
m_{t+1} &= \frac{1}{\theta_t} [\delta_t m_t + \varepsilon_t G + (1 - tax) \pi_t] - \pi_{t+1}. \\
S_{t+1} &= \lambda_t^d n' a \left(\frac{ab}{\alpha_t} \right)^{\frac{b}{1-b}} + S_t - \bar{Y}_t.
\end{aligned}$$

Classical Unemployment System

$$\begin{aligned}
\bar{L}_t &= L^d(1, \alpha_t). \\
\bar{Y}_t &= \frac{\alpha_t}{b} \bar{L}_t + S_t. \\
\lambda_t^s &= \frac{\bar{L}_t}{L^s}, \lambda_t^d = 1, \gamma_t^s = 1; \\
\text{if } \bar{Y}_t &\geq G + m_t, \text{ then } \gamma_t^d = \frac{\bar{Y}_t - m_t - G}{\tau(1 - tax)\pi_t + \tau\alpha_t \bar{L}_t}, \delta_t = \varepsilon_t = 1; \\
\text{if } G + m_t &> \bar{Y}_t \geq G, \text{ then } \gamma_t^d = 0, \delta_t = \frac{\bar{Y}_t - G}{m_t}, \varepsilon_t = 1; \\
\text{if } \bar{Y}_t < G, &\text{ then } \gamma_t^d = \delta_t = 0, \varepsilon_t = \frac{\bar{Y}_t}{G}. \\
\theta_t &= 1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3} \right). \\
\alpha_{t+1} &= \frac{1 - \nu_1(1 - \lambda_t^s)}{1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3} \right)} \alpha_t \\
\pi_{t+1} &= \frac{1}{\theta_t} (\bar{Y}_t - \alpha_t \bar{L}_t). \\
m_{t+1} &= \frac{1}{\theta_t} [\delta_t m_t + \varepsilon_t G + (1 - tax) \pi_t] - \pi_{t+1}. \\
S_{t+1} &= n' a \left(\frac{ab}{\alpha_t} \right)^{\frac{b}{1-b}} + S_t - \bar{Y}_t.
\end{aligned}$$

Underconsumption

$$\begin{aligned}
\bar{L}_t &= L^s. \\
\bar{Y}_t &= \tau(1 - tax) \pi_t + \tau\alpha_t L^s + m_t + G. \\
\lambda_t^s &= 1, \lambda_t^d = \frac{L^s}{L^d(\gamma_t^s, \alpha_t)} = \frac{(ab\gamma_t^s)^{1/(1-b)} L^s}{n' \alpha_t^{1/(1-b)}}; \\
\gamma_t^s &= \frac{\alpha_t}{ab} \left(\frac{\bar{L}_t}{n'} \right)^{1-b}, \gamma_t^d = 1, \delta_t = \varepsilon_t = 1. \\
\theta_t &= 1 - \mu_1(1 - \gamma_t^s). \\
\alpha_{t+1} &= \frac{1 + \nu_2(1 - \lambda_t^d)}{1 - \mu_1(1 - \gamma_t^s)} \alpha_t. \\
\pi_{t+1} &= \frac{1}{\theta_t} (\bar{Y}_t - \alpha_t \bar{L}_t). \\
m_{t+1} &= \frac{1}{\theta_t} [m_t + G + (1 - tax) \pi_t] - \pi_{t+1}. \\
S_{t+1} &= \lambda_t^d n' a \left(\frac{\gamma_t^s ab}{\alpha_t} \right)^{\frac{b}{1-b}} + S_t - \bar{Y}_t.
\end{aligned}$$